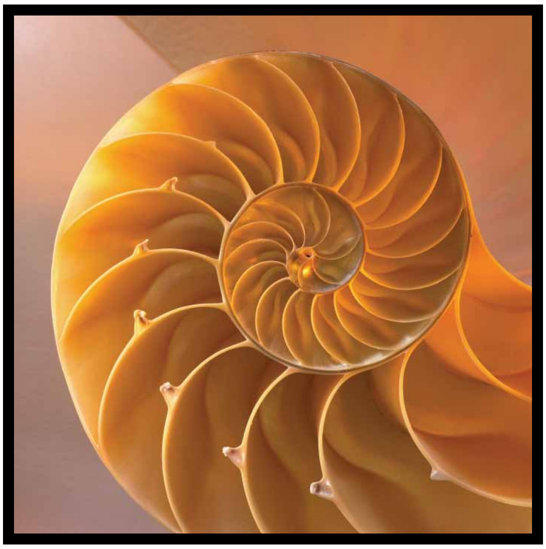
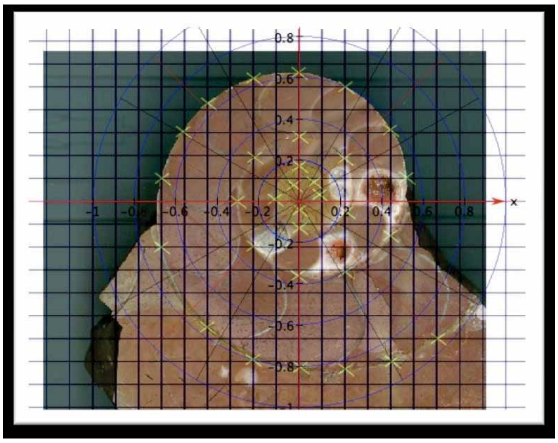
Spirals in Nature

When researching about mathematics in nature I found that certain spirals are found in shell shapes. The Nautilus is a marine mollusk with a spiral shell with partitions to create buoyancy. The adult mollusk can grow up to 25-30 centimetres across and the shell can withstand depths of up to 650 metres underwater. The chambers of the shell are separated but are interconnected via a tube running through them. The tube pours gas or liquid through the tube to move the creature around to sink or float respectively). The Nautilus shell's curves are logarithmic and equiangular with slightly different proportions to other spirals such as the Golden ratio.



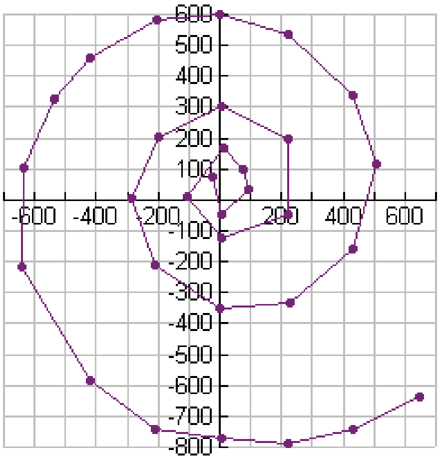
I have a fossil of an ammonite which has a spiral in it. I want to find out if this has an Archimedian spiral (a spiral described in polar coordinates by $r = a\theta$, where a is a constant), or a logarithmic spiral (a spiral described in polar coordinates by $r = ke^{c\theta}$ where $c = \cot\phi$), or something else altogether.

Here I have scanned the fossil and will attempt to model the spiral shown. In order to model the spiral, I have marked several points along it, according to the picture below and have computed the x- and y-coordinates.



There were 31 points. These are the coordinates and a plot of the coordinates to make sure I have the right shape.

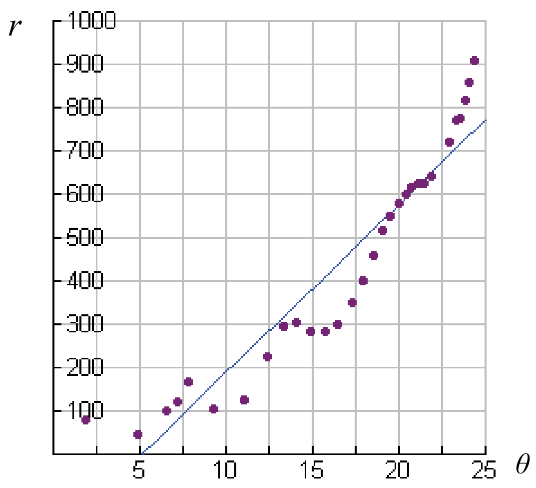
x	y
648	-639
434	-741
221	-786
7	-773
-212	-742
-421	-586
-639	-216
-635	105
-537	323
-421	461
-207	581
2	599
221	537
434	336
506	114
429	-158
225	-332
2	-349
-212	-212
-283	7
-198	203
7	305
221	198
221	-46.8
7	-126
-105	11
11	167
73	96
96	33
7	-47
-25	74



Now I want to see if this spiral can be described as $r = a\theta$, so I need to find the radius and angle for each point. I will find the angle in radians. The radius is easy: $r = \sqrt{x^2 + y^2}$. The angle is a bit more complicated because I want the point $(-25, 74)$ to have an angle of less than π, but I want the point $(-105, 11)$ to have an angle of nearly 3π because you can see from the plotting of the points that you have to go round more than a whole revolution to get there. $\arctan\left(\frac{y}{x}\right)$ gives an angle in the 1st or 4th quadrant so I worked out that if I know which quadrant a point is in and I keep the points all in order I can work out the angle by thinking about how many times I need to add π, so $\theta = \arctan\left(\frac{y}{x}\right) + n\pi$ where I define n in th following spreadsheet:

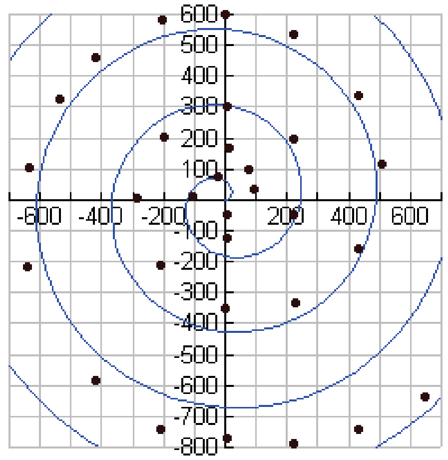
x	y	quadrant	n	θ	r	logθ	logr
648	-639	4	8	24.35433596	910.0687	1.386576	2.959074
434	-741	4	8	24.09177962	858.7415	1.381869	2.933862
221	-786	4	8	23.83603866	816.4784	1.377234	2.911945
7	-773	4	8	23.57100028	773.0317	1.372378	2.888197
-212	-742	3	7	23.28364524	771.6916	1.367051	2.887444
-421	-586	3	7	22.93895658	721.5518	1.360574	2.858268
-639	-216	3	7	22.31711851	674.5198	1.348638	2.828995
-635	105	2	7	21.82727704	643.6226	1.339	2.808631
-537	323	2	7	21.44963438	626.6562	1.33142	2.797029
-421	461	2	7	21.16042999	624.3092	1.325524	2.7954
-207	581	2	7	20.76261275	616.7739	1.317282	2.790126
2	599	1	6	20.41701336	599.0033	1.309992	2.777429
221	537	1	6	20.02993254	580.6979	1.301679	2.76395
434	336	1	6	19.50836196	548.8643	1.290221	2.739465
506	114	1	6	19.0711525	518.6829	1.280377	2.714902
429	-158	4	6	18.49667356	457.1706	1.267094	2.660078
225	-332	4	6	17.87436924	401.0598	1.252231	2.603209
2	-349	4	6	17.28449019	349.0057	1.237657	2.542833
-212	-212	3	5	16.49336143	299.8133	1.217309	2.476851
-283	7	2	5	15.68323333	283.0866	1.195436	2.451919
-198	203	2	5	14.91009692	283.5719	1.17348	2.452663
7	305	1	4	14.11422015	305.0803	1.149657	2.484414
221	198	1	4	13.29693121	296.7238	1.123751	2.472352
221	-46.8	4	4	12.35768885	225.901	1.091937	2.353918
7	-126	4	4	11.05107279	126.1943	1.043404	2.10104
-105	11	2	3	9.320396808	105.5746	0.969434	2.02356
11	167	1	2	7.788208383	167.3619	0.891438	2.223657
73	96	1	2	7.203847123	120.6027	0.857564	2.081357
96	33	1	2	6.614281384	101.5135	0.820483	2.006524
7	-47	4	2	4.860238346	47.51842	0.686658	1.676862
-25	74	2	1	1.896595442	78.1089	0.277975	1.892701

If the spiral follows an Archemedian spiral, $r = a\theta$, so plotting r against θ should give a straight line of gradient a intersecting the vertical axis at the origin. I have plotted $r$and θ and fitted a line of best fit by computer.

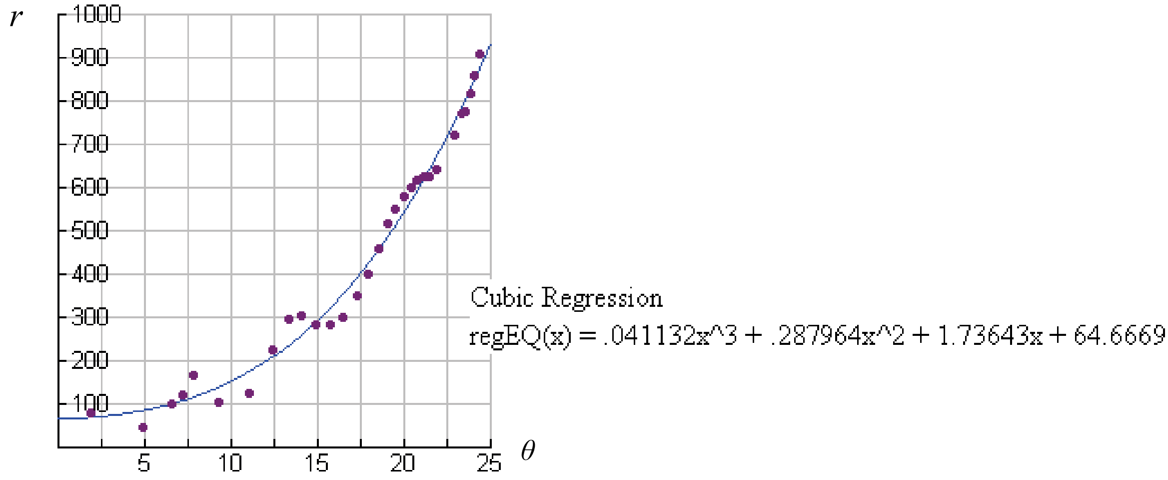


Linear Regression (ax+b)
regEQ(x) = 38.9346x + -201.183

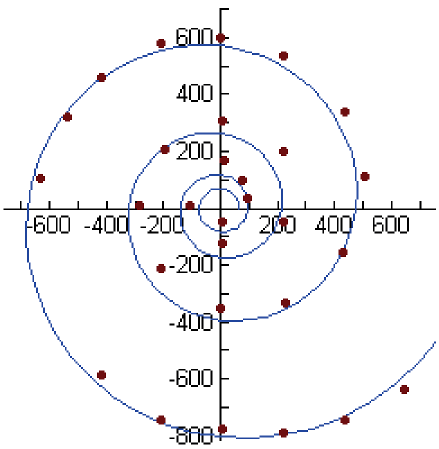
This doesn't look at all promising!
Anyway I'll try using a computer to draw a polar curve $r = 38.9346\,\theta$ on top of the data.



Well, it's a spiral, but it doesn't fit the points well.
I have looked at the graph of r against θ and used the computer to fit a quadratic curve to the points and a cubic curve to the points. The cubic seems to fit quite well. It looks like this:



I will try using the computer to draw $r = 0.041132\theta^3 + 0.287964\theta^2 + 1.73643\theta + 64.6669$ on top of the data.



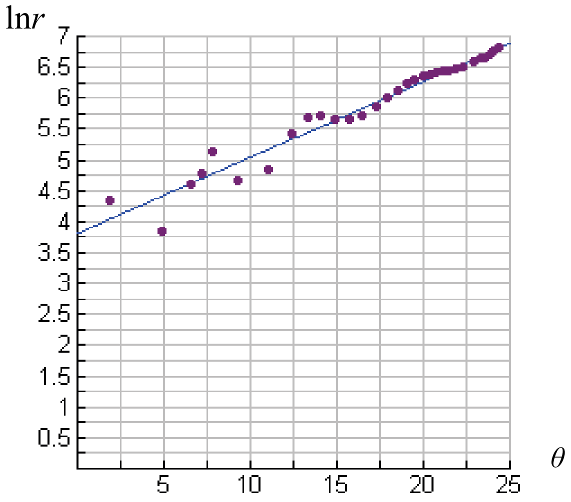
This does seem to follow the spiral more closely.

Now I will try fitting a logarithmic spiral. In the Encyclopaedia Britannica it said that the curve was of the form $r = ke^{c\theta}$ where $c = \cot\phi$. I think c is a constant, so

$$\begin{aligned}\ln r &= \ln ke^{c\theta} \\ &= \ln k + \ln e^{c\theta} \\ &= \ln k + c\theta \ln e \\ &= \ln k + c\theta\end{aligned}$$

So if I plot $\log r$ against θ I should get a straight line with gradient c and y-intercept $\ln k$.

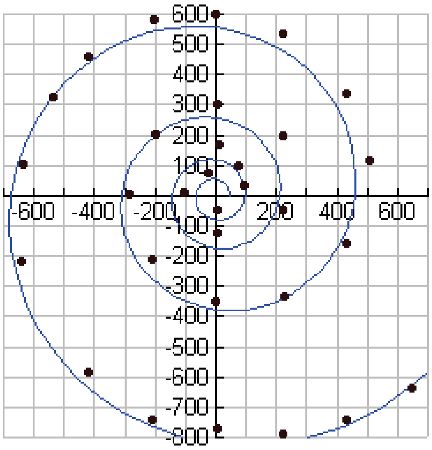
θ	r	lnr
24.35434	910.0687	6.81352
24.09178	858.7415	6.755468
23.83604	816.4784	6.705
23.571	773.0317	6.65032
23.28365	771.6916	6.648585
22.93896	721.5518	6.581404
22.31712	674.5198	6.514001
21.82728	643.6226	6.467112
21.44963	626.6562	6.440398
21.16043	624.3092	6.436646
20.76261	616.7739	6.424502
20.41701	599.0033	6.395267
20.02993	580.6979	6.364231
19.50836	548.8643	6.307851
19.07115	518.6829	6.251293
18.49667	457.1706	6.125057
17.87437	401.0598	5.994111
17.28449	349.0057	5.855088
16.49336	299.8133	5.70316
15.68323	283.0866	5.645753
14.9101	283.5719	5.647466
14.11422	305.0803	5.720575
13.29693	296.7238	5.692802
12.35769	225.901	5.420097
11.05107	126.1943	4.837823
9.320397	105.5746	4.659418
7.788208	167.3619	5.120158
7.203847	120.6027	4.792501
6.614281	101.5135	4.620192
4.860238	47.51842	3.861117
1.896595	78.1089	4.358104



Linear Regression (ax+b)
regEQ(x) = .123083x + 3.80866

That looks a bit more likely! I will try $c = 0.123083$, $\ln k = 3.80866$ so $k = e^{3.80866}$ $\quad k = 45.09$

So $r = 45.09 \times e^{0.123083\theta}$



I think this looks very good. The cubic example looked very good too, however, so now I am going to try to compare them. Every point on the spiral has a value of theta and a value of *r* and each model has an approximate value of r for that theta. I have decided to find the absolute error in each approximation of *r* and I will sum these errors for each of my models to see which has the least sum and hence is the closest to the real points on the spiral.

θ	r	38.9346θ	abs(r-approx)	cubic	abs(r-approx)	logarithmic	abs(r-approx)
24.35434	910.0687	948.22633	38.15765008	871.9249	38.14378905	903.4901656	6.578513127
24.09178	858.7415	938.0038	79.2622816	848.796	9.945556819	874.7595101	16.01798906
23.83604	816.4784	928.04663	111.5682173	826.6997	10.22125627	847.6532419	31.1748282
23.571	773.0317	917.72747	144.6957735	804.2457	31.21404107	820.4474975	47.4158035
23.28365	771.6916	906.53941	134.8477658	780.4084	8.716744071	791.9366489	20.24500065
22.93896	721.5518	893.1191	171.5672989	752.5031	30.95132975	759.0412207	37.48942073
22.31712	674.5198	868.90808	194.3882533	704.0279	29.50807324	703.1135427	28.59371351
21.82728	643.6226	849.8363	206.2137403	667.5018	23.87923796	661.97463	18.35206976
21.44963	626.6562	835.13293	208.4767293	640.3202	13.66404316	631.9092812	5.25307558
21.16043	624.3092	823.87288	199.5636593	620.0709	4.238321177	609.8113805	14.49783772
20.76261	616.7739	808.38402	191.6101577	593.0079	23.76599034	580.6714795	36.10238507
20.41701	599.0033	794.92825	195.9249096	570.23	28.77338712	556.4891914	42.51414745
20.02993	580.6979	779.85741	199.1595555	545.5137	35.18410793	530.5980037	50.09985234
19.50836	548.8643	759.55027	210.6859875	513.5152	35.34907515	497.605805	51.25847699
19.07115	518.6829	742.52769	223.8447468	487.8235	30.85943987	471.5358609	47.14708662
18.49667	457.1706	720.16059	262.9899399	455.5973	1.573385921	439.3456954	17.82495108
17.87437	401.0598	695.93142	294.8715709	422.601	21.54119205	406.9504697	5.890623833
17.28449	349.0057	672.96471	323.9589812	393.1081	44.10233203	378.4512999	29.44556931
16.49336	299.8133	642.16243	342.3491548	356.1887	56.37547441	343.337146	43.52387082
15.68323	283.0866	610.62042	327.5338571	321.3955	38.30890977	310.753471	27.66691183
14.9101	283.5719	580.51866	296.9467993	290.9142	7.342293638	282.5455996	1.026260809
14.11422	305.0803	549.53152	244.4511986	262.1922	42.88811306	256.1801867	48.9001306
13.29693	296.7238	517.7107	220.9869207	235.3722	61.35154229	231.6636828	65.06009451
12.35769	225.901	481.14167	255.2407206	207.7241	18.17688581	206.3723186	19.52863313
11.05107	126.1943	430.2691	304.0748041	174.5371	48.34277322	175.7143535	49.52005884
9.320397	105.5746	362.88592	257.3113034	139.1695	33.59484783	142.0022486	36.42763044

7.788208	167.3619	303.23078	135.8688947	115.0883	52.27362354	117.5961792	49.7657042
7.203847	120.6027	280.47891	159.8762528	107.4969	13.10570694	109.4350922	11.1675612
6.614281	101.5135	257.5244	156.0108541	100.6524	0.861137386	101.7751622	0.261616296
4.860238	47.51842	189.23144	141.7130184	84.63091	37.11249648	82.01262581	34.49420833
1.896595	78.1089	73.843185	4.265713442	69.27664	8.832254711	56.94575507	21.16314327
		sum	**6238.41671**	**sum**	**840.1973621**	**sum**	**914.4071688**

The Archemedian spiral had an absolute error sum of over 6000. The logarithmic spiral had an absolute error sum of just over 914 and the cubic spiral which I just made up because the points when I plotted r against θ looked as if they could follow a polynomial curve, has the smallest error sum of about 840. I expected the logarithmic spiral to fit best because I had read that Nautilus shells follow these curves, but that doesn't seem to be the case for my fossil.

It would be interesting to find other examples of ammonites from the many photographs available on the web, and to model their curves by cubic spirals and logarithmic spirals and to see which seem to fit more closely to find out whether the cubic shape is a good model for these spirals in general or whether it was a coincidence in this one case.

Bibliography

http://www.bsu.edu/web/math/exchange/01-01/allen.pdf

Britannica 2002 Deluxe Edition, Britannica.com Inc., 6 Jan. 2010

Murphy, Amanarth. "Maths of Nature and Nature of Maths Chapter 1." Scribd, 11 Jan. 2009. Web. 6 Jan. 2010. <http://www.scribd.com/doc/21990600/Maths-of-Nature-and-Nature-of-Maths-Chapter-1>.

Images

Chambered Nautilus. Digital image. *Inquiry By Design, Proportion*. 7 Nov. 2006. Web. 6 Jan. 2010. <http://blog.lib.umn.edu/myee/architecture/Nautilus%20Shell%202.gif>.

Shell. Digital image. *FH Perry Builder*. Web. 6 Jan. 2010. <http://www.fhperry.com/pages/relationships.html>.