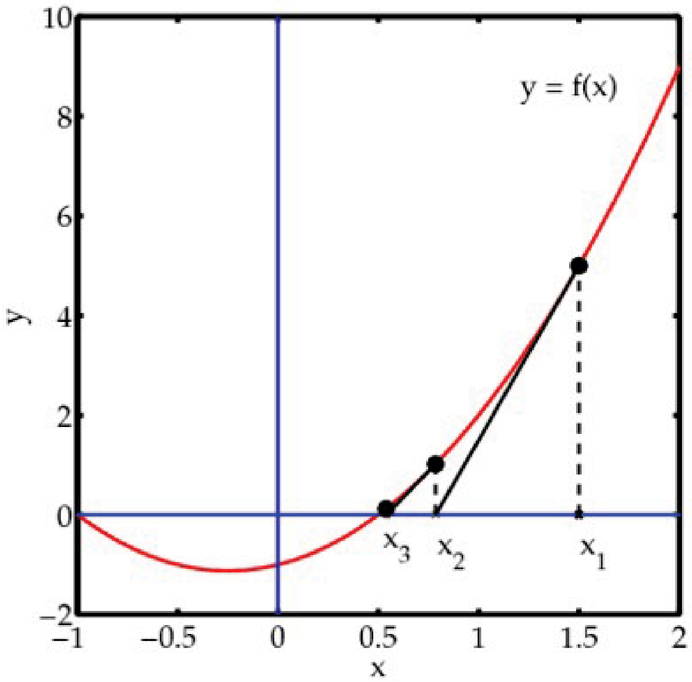
Maths Exploration

Newton-Raphson method



A Introduction includes aim and rationale

Rationale- For this project I chose to research and analyse the Newton-Raphson method, where calculus is used to approximate roots. I chose this topic because it looked extremely interesting and the idea of using calculus to approximate roots, seemed intriguing.

The aim of this exploration is to find out how to use the Newton-Raphson method, and in what situations this method is used

Explanation of the Newton-Raphson method

The Newton-Raphson or Newton's method is an iterative process to approximate roots. We know simple roots for rational numbers such as $\sqrt{4}$ or $\sqrt{9}$, but what about irrational numbers such as $\sqrt{3}$ or $\sqrt{5}$. This method was discovered in 1736 by Isaac Newton after being published in the 'Method of Fluxions', this method was also described by Joseph Raphson in 1690 in 'Analysis Aequationum'.

The Newton-Raphson Process:

In the Newton-Raphson process the following formula is used:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0$$

A, B *Lack of mathematical explanation of this terminology*

This is called the Newton-Raphson formula

How this formula was derived:

B *Student needs to be careful how the term "root" is used. Here it means the square root of "n", but the solution of f(x)=0 is also a root.*

The square root of a number (n) can be found by using the function:

$$f(x) = x^2 - n$$

The root of n is the value of x when f(x) = 0

B *No explanation of how an initial estimate is made*

The first thing you do when approximating a root is to make an initial estimate in terms of a positive integer and find the tangent of the function at that point.

Let the initial estimate = x_1

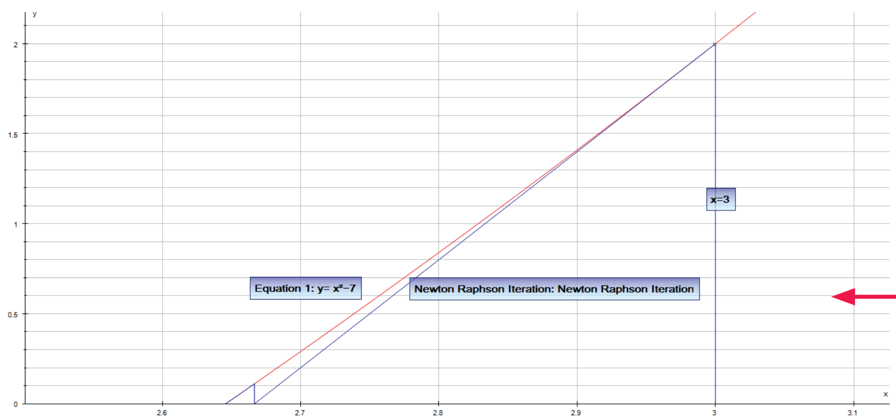
B *Poor use of subscripts throughout exploration*

The intercept of the tangent at x_1 with the x-axis will be closer to the root than the initial estimate. This intercept can be called x_2.

E *No explanation of how student knows it will be closer*

Calculating x_2 is done by studying the triangle bounded by the x-axis, the line $x=x_1$, and the tangent to the line at x_1. This only works when the gradient of the function at x_1 is not equal to zero.

An example of a triangle is shown in the following graph (where the square root of 7 is being estimated):



B *Graph could be larger and clearer, zooming in on the pertinent domain and range*

In this case $x_1=3$

This results in the following formula:

$$x_2 = x_1 - \frac{y_1}{gradient\ at\ x_1}$$

A Lack of explanation of where this comes from

Therefore

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

E Method is applied after this, but no evidence of understanding why the method works

Now that we have a better estimate, the same method can be used again to get closer to the answer. This time x_3 will be found from x_2 represented by the formula below:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

x_3 will be even closer to the root than x_2 was, and this procedure can be repeated an infinite number of times.

This is an explanation to the Newton-Raphson formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

A Not an explanation

Where n= the number of iterations

The more iterations that are carried out the more accurate the answer is. In practice the iteration is usually only repeated to x_{10} or less.

For example, if we use this method to approximate the value of root $\sqrt{7}$, if the equation $y = x^2 - 7$ is used.

C Good demonstration of applying unfamiliar mathematics

The formula for this function is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^{\ 2} - 7}{2x_n}$$

E Differentiation within syllabus

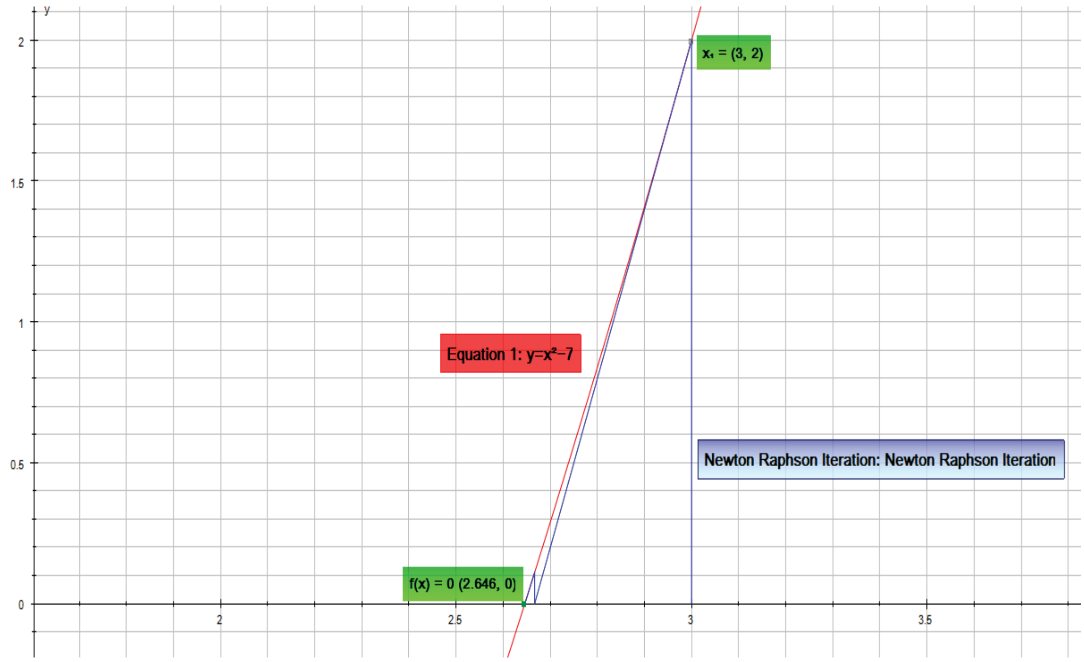
Using the Newton-Raphson formula for this function the following table showing the value of x for different numbers of iterations:

x_1	3.0000000000
x_2	2.6666666667
x_3	2.6458333333
x_4	2.6457513123
x_5	2.6457513111
x_6	2.6457513111
x_7	2.6457513111
x_8	2.6457513111
x_9	2.6457513111
x_{10}	2.6457513111

A Lack of detailed explanation

The table shows that this method will only need a few iterations and you already have the value of x to 3 decimal places and after the fifth iteration you have the values of x up to 10 decimal places.

Graph showing Newton-Raphson method for $y = x^2 - 7$, when approximating $\sqrt{7}$



A Graph does not aid coherence and adds nothing to previous graph.

I will now do the same for $\sqrt{3}$, and I'll use the equation $y = x^2 - 3$

The formula for this function is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

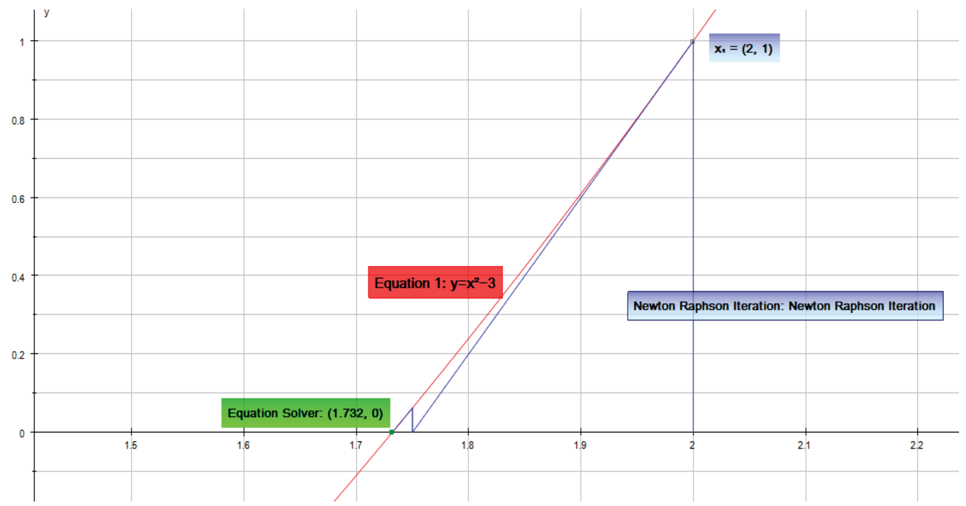
$$x_{n+1} = x_n - \frac{x_n{}^2 - 3}{2x_n}$$

Using the Newton-Raphson formula for this function the following table showing the value of x for different numbers of iterations:

x_1	2.0000000000
x_2	1.7500000000
x_3	1.7321428571
x_4	1.7320508100
x_5	1.7320508076
x_6	1.7320508076
x_7	1.7320508076
x_8	1.7320508076
x_9	1.7320508076
x_{10}	1.7320508076

The table again shows that this method will only need a few iterations and you already have the value of x to 3 decimal places and after the fifth iteration you have the values of x up to 10 decimal places.

.Graph showing Newton-Raphson method for $y = x^2 - 3$, when approximating $\sqrt{3}$



Using a spreadsheet, I repeated the same method for all roots $\sqrt{x}$ where x $\in Z^{+}$ in the domain $1 \le x \le 7$, and a table of iterations was produced:

B *Good use of notation*

	1	2	3	4	5	6	7
x_1	1.0000000000	1.5000000000	2.0000000000	2.0000000000	2.5000000000	2.5000000000	3.0000000000
x_2	1.0000000000	1.4166666667	1.7500000000	2.0000000000	2.2500000000	2.4500000000	2.6666666667
x_3	1.0000000000	1.4142156863	1.7321428571	2.0000000000	2.2361111111	2.4494897959	2.6458333333
x_4	1.0000000000	1.4142135624	1.7320508100	2.0000000000	2.2360679779	2.4494897428	2.6457513123
x_5	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
x_6	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
x_7	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
x_8	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
x_9	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
x_{10}	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111

The table shows that after the first few iterations you get a very precise value of x

Although the main function of this method is to approximate roots it can also be used for non-linear equations.

C, E *Student could have mentioned other roots, such as cube roots.*

For example the Newton-Raphson method can be used to find when tan(x) = x or cos(x)= x

Where f(x) = x – tan x or f(x) = x – cos x

The Newton Raphson formula can easily be adjusted for these equations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

When $f(x) = x - tan\, x$

C *Student created own example, but did not follow through to complete application with graphical analysis to see how the process worked for this case.*

Becomes:

$$x_{n+1} = x_n - \frac{x_n - tan(x_n)}{1 - sec^2(x_n)}$$

And when $f(x) = x - cos\,x$

D Opportunity for reflection on any special implications to non-linear functions

Becomes:

$$x_{n+1} = x_n - \frac{x_n - cos(x_n)}{1 + sin(x_n)}$$

Reflection:

Evaluation and conclusion

This was a very interesting exploration, as at the start I had no idea what the formula mean and the logic behind it. The idea of using the x-intercept of the gradient to get closer and closer to the correct value of x was amazing. Up to now, I always had a calculator to find the root or zero of a number. If I didn't I would just leave it as the $\sqrt{3}$ or $\sqrt{5}$. Although, this method may not be practical to use on a maths test, it has quite a few advantages. This method gives us the chance to approximate roots up to thousands of decimal places. Even after the first five iterations the value of x is given to 10 decimal places. When finding the root of x, it is quite easy to work out x_2 in your head, which is very close to the precise value. Furthermore, this formula is implemented in technology such as autograph, where Newton-Raphson Iteration is very easy to use and find, as well as by using spreadsheets where it is simple to implement the formula for any number of iterations. Nevertheless, it's easy to find the zero or root by finding the x-intercept, which is much easier than using the Newton-Raphson method on the calculator or any graphing program. This method would be useful though in any field involving zeros or roots where a high-level of precision is needed.

B Confused use of "root" and "zero"

D Superficial reflection with no depth

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